Sampling over spiraling curves

Philippe JAMING, Felipe NEGREIRA  
Univ. Bordeaux, IMB, UMR 5251  
F-33400 Talence, France  
CNRS, IMB, UMR 5251  
F-33400 Talence, France  
philippe.jaming@math.u-bordeaux.fr  
felipe.negreira@math.u-bordeaux.fr

José Luis ROMERO  
Faculty of Mathematics, University of Vienna  
Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria  
Acoustics Research Institute, Austrian Academy of Science  
Wohllebengasse 12-14, 1040 Vienna, Austria  
jose.luis.romero@univie.ac.at  
jlromero@kfs.oeaw.ac.at

Abstract—We present our recent work on sampling along spiral-like curves [6], and discuss the main techniques. As a first result we give a sharp density condition for sampling on spirals in terms of the separation between consecutive branches. We then further show that, below this sharp density condition for sampling on spirals in terms of the separation curves [6], and discuss the main techniques. As a first result we give a Hausdorff (length) measure. Here,  

$$\mathcal{H}(\Gamma) := \inf \{ \| \hat{f} \|_{L^2(\Gamma)} : f \in \mathcal{F}(\Omega), \| f \|_{L^2} = 1 \}.$$  

Equivalently, one aims to reconstruct a function that is supported from samples of its Fourier transform taken along a curve. This problem is relevant, for example, in magnetic resonance imaging (MRI), where moving sensors capture the anatomy and physiology of a patient.

For pointwise sampling the key quantity is the Beurling density of a set [1], which measures the average number of samples per unit volume. When dealing with continuous trajectories, however, a more meaningful metric is the average length covered by a curve [5], [8], as a proxy for scanning times. On the other hand, the understanding of sampling trajectories in terms of their length is more subtle than the discrete case, and it is less clear what can be said in full generality [5]. Nonetheless, for some particular cases - such as parallel lines - it is possible to give a complete characterization of the sampling problem. Here we give a solution for a family curves that we call spiraling. The main two examples of such curves are the collection of concentric circles

$$O^0 := \{(x, y) : x^2 + y^2 = \eta^2 k^2, k \in \mathbb{N}\},$$  

and the Archimedes spiral

$$A^0 := \{(\eta \cos 2\pi \theta, \eta \sin 2\pi \theta) : \theta \geq 0\},$$  

see Figures 1 and 2 below.

Our first main result reads as follows.

Theorem 1 (6]). Let $$\Omega \subset \mathbb{R}^2$$ be a convex centered symmetric body.  
(i) If diam$$\Omega$$ < 1, then the Archimedes spiral $$A^0$$ and the collection of concentric circles $$O^0$$ are sampling trajectories for $$\mathcal{P}^2(\Omega)$$.  
(ii) If diam$$\Omega$$ > 1, then neither the Archimedes spiral $$A^0$$ nor the collection of concentric circles $$O^0$$ are sampling trajectories for $$\mathcal{P}^2(\Omega)$$.

In the reference case of the unit square $$\Omega = [-1/2, 1/2]^2$$, Theorem 1 tells us that the critical value for reconstruction over $$A^0$$ or $$O^0$$ - i.e. their Nyquist rate - is $$\eta = \sqrt{2}/2$$.

We then consider slightly less dense spirals, and restrict the reconstruction problem to functions that are compactly represented in certain dictionaries. In many modern sampling schemes, such undersampling is expected to be possible because many signals of interest are highly compressible [7]. In this direction, we obtain a result for functions obeying a variation bound:

$$\mathcal{F}(W) := \{ f \in L^2([-1/2, 1/2]^2) : \text{var}(f) \leq W \},$$  

where $$\text{var}(f) := \sup \{ f \text{div} h : h \in C^2_\sigma, \| h \|_{\infty} \leq 1 \}$$. Here, the resolution parameter $$W > 0$$ essentially controls the number of active wavelet coefficients [4].

Theorem 2. Let $$\eta = (1 + \varepsilon)\sqrt{2}/2$$ with $$\varepsilon \in (0, 1)$$, and $$\Gamma = A^0$$ or $$\Gamma = O^0$$. For $$W > 0$$ set the stability margin

$$A(\Gamma, W) := \inf \{ \| \hat{f} \|_{L^2(\Omega)} : f \in \mathcal{F}(\Omega), \| f \|_{L^2} = 1 \}.$$  

Then there exists $$C > 0$$ such that

$$A(\Gamma, W) \leq C(\varepsilon W)^{-1/2}(\ln^2(\varepsilon W) + 1), \forall W > 0.$$  

Roughly speaking, Theorem 2 says that when undersampling by a small factor $$1 - \varepsilon$$ one can only recover functions up to resolution $$W \approx \varepsilon^{-1}$$ with a stable condition number.

Further, exploring the precise relation given in [4], we can reformulate this result explicitly in terms of the Haar wavelet. Indeed, let $$\Sigma_{N,J}$$ be the class of functions on $$[-1/2, 1/2]^2$$ with $$N$$ non-zero Haar coefficients, all of them taken with scale at most $$J$$. We then have the following estimate.

Theorem 3. Let $$\eta = (1 + \varepsilon)\sqrt{2}/2$$ with $$\varepsilon \in (0, 1)$$, and $$\Gamma = A^0$$ or $$\Gamma = O^0$$. Then for $$N \geq 1$$,

$$A_{N,J}(\Gamma) \leq K N^{-1/6} \varepsilon^{-1/3} N^{1/3},$$  

where $$J = \ln(\varepsilon^{-1} N)$$, and $$K > 0$$ is a universal constant.

Informally, Theorem 3 says that when undersampling by a small factor $$1 - \varepsilon$$, one can recover at most $$N \approx \varepsilon^{-6}$$ Haar coefficients with a stable condition number. This complements related results that limit the wavelet-sparsity of discrete signals that can be sampled on unions of parallel lines [2]. These results underscore the need for a certain level of randomness in sampling trajectories, and for the exploitation of finer multi-scale models that apply to generic signals [3].
REFERENCES


