Flexible sparse regularization with general non-convex penalties

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Abstract—We investigate regularization of linear inverse problems by generalized Tikhonov regularization that promotes sparsity. We are interested in penalties on sequence spaces where each coordinate is regularized by an individual penalty function and specifically in the case where these functions are non-convex.

I. INTRODUCTION

The aim of this work is to point out a unifying framework for sparsity regularization of linear inverse problems by using quite general flexible penalties that regularize different indices with a different sparsity promoting term as it has been proposed in [10] for the case of variable exponent regularizers. Let $A: \ell^2 \to Y$ be a linear and bounded operator which is also $\ell^1$-weak*-weak sequentially continuous. Then for any $\alpha > 0$, the Tikhonov functional defined in (2) has at least one minimizer $x_\alpha$.

Additionally, even without the additional continuity assumption, all minimizers are sparse.

Proposition 4 (Regularization): Let $A: \ell^2 \to Y$ be a linear and bounded operator which is also $\ell^1$-weak*-weak sequentially continuous and that the functions $\phi_k$ are as in the previous proposition. If there is a solution of $Ax = y$, then there is a subsequence of the sequence of minimizers $(x_n)$ of the Tikhonov minimization problems which converges to a solution $\bar{x} \in \arg\min_{Ax=y} \phi(x)$ in the sense that $\phi(x_n - \bar{x}) \to 0$ as $\alpha \to 0$ (and by Proposition 2 also in the $\ell^1$-norm).

III. EXAMPLES

Some examples of functionals $\phi$ which are inspired by slightly different functions in [1], [3], [12] are

1) $\phi(x) = \sum_{k \in \mathbb{N}} |x_k|^p_k, \quad p_k \in (0, 2)
2) $\phi(x) = \sum_{k \in \mathbb{N}} \log(|x_k|^p_k + 1), \quad p_k \in (0, 2)
3) $\phi(x) = \sum_{k \in \mathbb{N}} \log(|x_k| + |x_k|^p_k), \quad p_k \in (0, 1)
4) $\phi(x) = \sum_{k \in \mathbb{N}} (|x_k| + |x_k|^p_k)^{q_k}, \quad p_k, q_k \in (0, 1)
5) $\phi(x) = \sum_{k \in \mathbb{N}} |x_k|^p \log(|x_k| + 1), \quad p_k \in (0, 1)
6) $\phi(x) = \sum_{k \in \mathbb{N}} |x_k| (\log(|x_k| + 1))^{p_k}, \quad p_k \in (0, 1).

One can check that the first four examples of functionals $\phi$ for $p_k \in (0, 1)$ satisfy all the assumptions mentioned above and are not convex, while the functionals in the last two examples do not satisfy (a), as the components $\phi_k$ behave like the convex function $t \mapsto t^{p_k+1}$ for $p + 1 > 1$ when $t$ is small enough. Moreover, the minimizers $x_\alpha$ are not necessarily sparse in the latter case. Illustrations for the first three functions $\phi$ are in Figures 1, 2, and 3.

Numerical experiments for the various sparsity settings proposed here, e.g. using iteratively reweighting algorithms as in [12], will be the subject of another work.

\[\phi_k(t) \geq \frac{c}{t^{1/p_k+1}}.\]

Assumption (b): The set $\{t \geq 0 : \phi_k(t) \leq M, \forall k \in \mathbb{N}\}$ is bounded, for any $M > 0$, .
Fig. 1. The function $x \mapsto |x|^p$ from 1) for $p = 0.2, 0.4, 0.6, 0.8, 1$.

Fig. 2. The function $x \mapsto \log(|x|^p + 1)$ from 2) for $p = 0.2, 0.4, 0.6, 0.8, 1$.

Fig. 3. The function $x \mapsto \log(x + 1) + |x|^p$ from 3) for $p = 0.2, 0.4, 0.6, 0.8, 1$.

REFERENCES