Abstract — Total variation (TV) norm has been widely used in the context of compressed sensing. In this paper, given multiple measurement vectors (MMVs), we study the alternating direction method of multipliers (ADMM) algorithm to solve the multiple TV (MTV) norm minimization problem. In ADMM-MTV, we derive the closed-form solutions of all subproblems to ease implementation. Moreover, ADMM-MTV is able to reconstruct all image signals simultaneously, which is more efficient than the traditional TV norm minimization problem that conducts sparse recovery individually. In addition to computation complexity analysis and comparison, we also verify that the MTV norm outperforms TV norm individually. In the literature, in order to improve TV norm optimization, more complicated regularization functions mixed with TV norm [4], [6], [7] have been proposed. By introducing multiple measurement vectors (MMVs), ℓ_2,1-norm optimization [8], [9], an MMVs extension of ℓ_1-norm, has been proposed. The researches of MMVs extension of TV norm [10], [11], [12], [13], [14], [15], [16], [17] also have received considerable attention but with different definitions regarding the extended TV norm.

I. INTRODUCTION

Compressed sensing [1], [2], [3] has been studied extensively over the past few years. According to the prior knowledge of image structure, ℓ_1-norm and Total Variation (TV) norm optimizations [4], [5] are the most common choice for solving sparse signal recovery.

In the literature, in order to improve TV norm optimization, more complicated regularization functions mixed with TV norm [4], [6], [7] have been proposed. By introducing multiple measurement vectors (MMVs), ℓ_2,1-norm optimization [8], [9], an MMVs extension of ℓ_1-norm, has been proposed. The researches of MMVs extension of TV norm [10], [11], [12], [13], [14], [15], [16], [17] also have received considerable attention but with different definitions regarding the extended TV norm.

A. Problem Definition

Given l (> 1) signals, each of which is of size n × n and has sparse gradients. Let \( s_i \in \mathbb{R}^{n^2} \), \( i = 1, 2, \ldots, l \), be the corresponding vectorized counterparts. Let \( S = [s_1, s_2, \ldots, s_l] \in \mathbb{R}^{n^2 \times l} \) and let \( Y = [y_1, y_2, \ldots, y_l] \in \mathbb{R}^{m \times nl} \) be the measurement vectors corresponding to \( S \), which satisfies \( Y = AS \), where \( A \in \mathbb{R}^{m \times n^2} \) denotes the sensing matrix. Similar to TV norm optimization, we reconstruct \( S \) from \( Y \) and \( A \) by the following optimization problem:

\[
\text{(MTV)} \quad \min_S \|S\|_{MTV} \quad \text{s.t.} \quad Y = AS,
\]

where \( \|S\|_{MTV} = \sum_{i=1}^{n^2} \sqrt{\sum_{j=1}^{m} \|D_i s_j\|^2} \) with \( D_i \in \mathbb{R}^{2 \times n^2} \) as illustrated in Fig. 1, being the first-order finite differences of \( s \) at pixel \( i \) along both the horizontal and vertical directions. More precisely, the first row of \( D_i \) is \( e_i - e_{i+1} \), where \( e_i \) denotes the \( i^{th} \) standard basis vector; and the second row of \( D_i \) is \( e_{i+1} - e_i \).

The concept of MTV norm is inspired by ℓ_2,1-norm in that \( D_i \) represents the gradient operator and all \( D_i s_j \) should share the common supports, which refer to the co-sparsity property. Our goal is to derive an alternating direction method of multipliers (ADMM) algorithm of (MTV) problem.

II. MAIN RESULTS

Let \( \|\cdot\|_F \) be a Frobenius norm, let \( \langle \cdot, \cdot \rangle_F \) be a Frobenius inner product, let \( A^+ \) be a Moore-Penrose pseudo-inverse of matrix \( A \), and let \( [n_1 : n_2] \) denote a set of integers, including \( n_1, n_1 + 1, \cdots, n_2 \). We let \( G = \left( \mu A^T A + \beta \left( \sum_{i=1}^{n_2} D_i^2 D_i \right) \right)^{-1} \) for short.

A. ADMM Algorithm of (TV)

It is known that the ADMM Algorithm of (TV) (ADMM-TV) can be summarized as the following 1st subproblem, 2nd subproblem, and dual variable update, respectively:

\[
\begin{align*}
  z^{k+1} &= \max \left\{ \|D_i s^k + \frac{1}{2} \lambda_i^k\|_2^2 - \frac{1}{\beta_i} \langle \cdot, 0 \rangle \right\}, \quad i \in [1 : n^2], \\
  s^{k+1} &= G \left( \mu A^T y - \sum_{i=1}^{n^2} D_i^T \left( \lambda_i^k - \beta_i z_i^{k+1} \right) \right), \\
  \lambda_i^{k+1} &= \lambda_i^k - \gamma \beta_i (z_i^{k+1} - D_i s^{k+1}).
\end{align*}
\]

B. ADMM Algorithm of (MTV)

In this paper, we derive the ADMM algorithm of (MTV) as:

\[
\begin{align*}
  Z^{k+1} &= \max \left\{ \|D_i S^k + \frac{1}{2} \Lambda_i^k\|_F^2 - \frac{1}{\beta_i} \langle \cdot, 0 \rangle \right\}, \quad i \in [1 : n^2], \\
  S^{k+1} &= G \left( \mu A^T Y - \sum_{i=1}^{n^2} D_i^T \left( \Lambda_i^k - \beta_2 Z_i^{k+1} \right) \right), \\
  \Lambda_i^{k+1} &= \Lambda_i^k - \gamma \beta_i (Z_i^{k+1} - D_i S^{k+1}).
\end{align*}
\]

III. VERIFICATION

In the experiments, two representative image sequences: Basketball video (left in Fig. 2) and MRI image sequence (right in Fig. 2) with a frame size of 512 × 512 pixels were adopted. The measurement rate (MR) is defined to be \( m/n^2 \).

For basketball video, we adopted block-based sensing and reconstruction by dividing each frame into 16 small blocks with a size of \( n \times n \), where \( n = 128 \). Block signal recovery was performed via (TV) and (MTV). In this paper, \( s_1, s_2, \cdots, s_l \) are the wavelet coefficients of the original video by letting \( A = \Phi \Psi \), where \( \Phi \) is the measurement matrix chosen as a Gaussian random matrix of size \( m \times n^2 \) and the dictionary \( \Psi \) is an inverse wavelet. For MRI image sequences, we divided each MRI image into 256 small blocks with a size of 32 × 32 pixels for block-based sensing and recovery. For each block, due to the special structure of MRI images, we let \( s_j \) be the original images, and let \( A \) be random partial Fourier matrix. The parameters for both (TV) and (MTV) were chosen as \( \beta = 0.0002, \mu = 1, \) and \( \gamma = 1 \). As shown in Fig. 3 and Table I, PSNRs of (MTV) are stably higher than those of (TV).

Moreover, Table II listed the complexity of ADMM-TV* and ADMM-MTV, respectively. On the other hand, it is worth mentioning that, if we stack all the images to form a vector and using an appropriate first-order difference matrix \( \bar{D} \), the (MTV) problem could also be formulated and solved by the algorithm ADMM-TV, which we call modified ADMM-TV (M-ADMM-TV). But due to the stacking, the vector length will be \( n^2 l \) and the sensing matrix will become a sparse block diagonal matrix with size \( m l \times n^2 l \). This leads to very high complexity if we do not optimize M-ADMM-TV by the sparse structure of sensing matrix. It is observed that the cost of ADMM-MTV is comparable with that of M-ADMM-TV with optimization in Table II.

*Since ADMM-TV is an SMV algorithm, we need to repeat \( l \) times for reconstruct \( s_i \)’s individually.
**REFERENCES**


