Compressive Phase Retrieval of Structured Signals

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Abstract—Compressive phase retrieval is the problem of recovering a structured vector $x \in \mathbb{C}^n$ from its phaseless linear measurements. A compression algorithm aims to represent structured signals with as few bits as possible. As a result of extensive research devoted to compression algorithms, in many signal classes, compression algorithms are capable of employing sophisticated structures in signals and compress them efficiently. This raises the following important question: Can a compression algorithm be used for the compressive phase retrieval problem? To address this question, Compressive Phase Retrieval (COPER) optimization is proposed. COPER can employ any compression algorithm, without having any prior information about it, and perform phase retrieval. For a family of compression codes with rate-distortion function denoted by $r(\delta)$, in the noiseless setting, COPER is shown to require slightly more than $\kappa = \lim_{\delta \to 0} \frac{\log(1/\delta)}{\log \log(1/\delta)}$ observations for an almost accurate recovery of $x$. COPER is based on combinatorial computationally-demanding optimization problem. To efficiently approximate the solution of COPER, we propose an iterative algorithm namely, gradient descent for COPER (GD-COPER). We will prove that under some mild conditions on the initialization, if the number of phaseless measurements is larger than $C\kappa^2 \log^2 n$, where $C$ is a fixed number, then GD-COPER obtains an accurate estimate of $x$ in a polynomial time. In addition, simulation on real data confirms the theoretical results we obtain in this work.

I. MOTIVATION

Consider the problem of recovering $x \in \mathcal{Q}$, where $\mathcal{Q}$ denotes a compact subset of $\mathbb{C}^n$, from $m$ noisy phase-less linear observations

$$y = |Ax| + \epsilon,$$

where $\epsilon \in \mathbb{R}^m$ and $A \in \mathbb{C}^{m \times n}$ denote the measurement noise, and the sensing matrix, respectively. Here, $| \cdot |$ denotes the element-wise absolute value operator. There has been a lot of work on this subject. \cite{1-8} are some of the researches that investigated phase retrieval. In this work, we further assume that the class of signals $\mathcal{Q}$ is “structured”, but instead of the set $\mathcal{Q}$, or its underlying structure, for recovering $x$ from $y$, we have access to a compression code that takes advantage of the structure of signals in $\mathcal{Q}$ to compress them efficiently. For instance, consider the class of images or videos for which compression algorithms, such as JPEG2000 or MPEG4, capture complicated structures within such signals and encode them efficiently. Employing such structures in a phase retrieval algorithm can reduce the number of measurements or equivalently increase the quality of the recovered signals. This raises the following questions:

1) Is it possible to use a given compression algorithm for the set $\mathcal{Q}$ for the recovery of $x \in \mathcal{Q}$ from its undersampled set of phaseless observations?

2) If so, what is the required number of observations?

3) How many observations do we need to recover $x$ from its undersampled set of phaseless observations in polynomial time?

Answering these questions enables us to employ the structures that are employed by the state-of-the-art compression algorithms, such as JPEG2000 or MPEG4, to improve the quality of the recovered signals or decrease the required number of measurements for a given quality. Furthermore, if the image or video compression communities develop new compression algorithms that are capable of employing new and more complex structures, then the framework we develop in this paper leads to a phase retrieval algorithm, with no extra effort, that takes advantage of such complicated structures.

In this work, we address the mentioned questions. In the following we briefly review some of our results.

II. ACHIEVEMENTS

A rate-$r$ compression code for compact set $\mathcal{Q}$ is defined through an encoder $\mathcal{E}_r : \mathbb{Q} \rightarrow \{1, \ldots, 2^r\}$ and a decoder $\mathcal{D}_r : \{1, \ldots, 2^r\} \rightarrow \mathbb{C}^n$. Compression code $(\mathcal{E}_r, \mathcal{D}_r)$ is associated with distortion $\delta(r) \triangleq \sup_{x \in \mathcal{Q}} |x - D(\mathcal{E}_r(x))|$ and codebook $\mathcal{C}_r \triangleq \{D_r(\mathcal{E}_r(x)) : x \in \mathcal{Q}\}$. Given $x, c \in \mathbb{C}^n$, define their distance with respect to sensing matrix $A$ as

$$d(x, c) \triangleq \sum_{k=1}^{m} (a_k^* x - a_k^* c)^2 = \sum_{k=1}^{m} (a_k^* (xx^* - cc^*)a_k)^2,$$

where $a_k^*$ denotes the $k$th row of $A$.

COPER recovers $x$ from its phase-less measurements $y$ by solving the following non-convex optimization:

$$\hat{x} = \arg\min_{c \in \mathcal{C}_r} d(x, c).$$

COPER is clearly a non-convex optimization which potentially requires performing an exhaustive search over $2^r$ codewords. To derive a computationally efficient compression-based recovery method, we propose GD-COPER, which operates as follows. Start from $z_0$, such that $||x - z_0|| < ||x||$. For $t = 1, 2, \ldots$

$$z_{t+1} \triangleq D_r(\mathcal{E}_r(z_t - \mu\nabla d(z_t))).$$

GD-COPER is a projective gradient descent algorithm in which $\mathcal{D}_r \circ \mathcal{E}_r$ plays the role of projection. We obtain the following results in the noiseless setting, i.e. $\epsilon = 0$. Recall $\kappa = \lim_{\delta \to 0} \frac{\log(1/\delta)}{\log \log(1/\delta)}$. We show that with $(1 + \eta)\kappa$ observations, where $\eta > 0$ is arbitrary, COPER’s solution $\hat{x}$, gets arbitrarily close to $x$ as rate of the compression grows. Moreover, having $C\kappa^2 \log^2 n$ measurements, GD-COPER converges to $\hat{x}$ at an exponential rate. Note that $\kappa$ is the information theoretical lower bound for number of phase-less observations with which one can solve the phase retrieval problem and to the best of our knowledge, all known algorithms with polynomial run time need at least $O(\kappa^2)$ observations for a satisfactory recovery.

More details about the proposed recovery methods, i.e., COPER and GD-COPER, the proofs of the main theorems and some preliminary simulation results could be found in \cite{12}.
REFERENCES


